On Galilean Isometries

Christian DUVAL*

Centre de Physique Théorique CNRS, Case 907
F–13288 MARSEILLE Cedex 9, FRANCE
(Christian.Duval@cpt.univ-mrs.fr)

We introduce three nested Lie algebras of infinitesimal 'isometries' of a Galilei spacetime structure which play the rôle of the algebra of Killing vector fields of a relativistic Lorentz spacetime. Non trivial extensions of these Lie algebras arise naturally from the consideration of Newton-Cartan-Bargmann automorphisms.

0. Introduction

Quite recently, Carter and Khalatnikov [**CK**] have pointed out that a geometric four-dimensional formulation of the non relativistic Landau theory of perfect superfluid dynamics should involve not only Galilei covariance but also, more significantly as far as gravitational effects are concerned, covariance under a larger symmetry group which they call the *Milne group* after Milne's pioneering work in Newtonian cosmology [**Mi**].

The purpose of this note is to show that the degeneracy of the Galilei 'metric' [**Ca**,**Tra**,**Kü**] allows for a certain flexibility in the definition of spacetime 'isometries'. More precisely, *three* different nested Lie algebras of spacetime vector fields naturally arise as candidates for Newton-Cartan symmetry algebras, one of them being the newly highlighted Milne algebra. This is reviewed in section 1 in a simple algebraic way.

Section 2 is devoted to a detailed study of the various extensions of these isometries in the framework of Newton-Cartan-Bargmann structures associated with a Newtonian principle of general covariance that goes back to Cartan. These nested Lie algebras actually embody the *Bargmann* algebra which generates the fundamental symmetry group of massive, either classical or quantum, non relativistic isolated systems.

* CPT-CNRS and Université d'Aix-Marseille II

1. Coriolis, Galilei and Milne algebras

Let us recall that a *Newton-Cartan* (NC) *structure* [**Kü**] for spacetime, $(M, \gamma, \theta, \Gamma)$, consists of a smooth manifold M of dimension $n+1$, a 2-contravariant semi-positive symmetric tensor field $\gamma = \gamma^{ab}\,\partial_a \otimes \partial_b$ $(a, b = 0, 1, \ldots, n)$ whose kernel is spanned by the *time* 1-form $\theta = \theta_a\,dx^a$; also Γ is a torsion-free linear connection compatible with γ and θ. Now such a connection is far from being uniquely determined by the *Galilei structure* (M, γ, θ), therefore *Newtonian connections* are furthermore subject to the nontrivial symmetry of the curvature $R_{a\ c}^{\ b\ d} = R_{c\ a}^{\ d\ b}$ (where $R_{a\ c}^{\ b\ d} \equiv \gamma^{bk}\,R_{akc}{}^{d}$) which may be thought of as part of the gravitational field equations [**DK**].

The *standard* example of a NC structure is given by $M \subseteq \mathbf{R} \times \mathbf{R}^n$ together with $\gamma = \sum_{A=1}^{n} \partial_A \otimes \partial_A$ and $\theta = dx^0$; the nonzero components of the connection $\Gamma^A_{00} = \partial_A \phi$ $(A = 1, \ldots, n)$ accomodate the Newtonian scalar potential ϕ.

The *flat* NC structure corresponds to the trivial case $\Gamma_{ab}{}^{c} \equiv 0$.

i) *Coriolis algebra*

The vector fields $X = X^a\,\partial_a$ on M that satisfy

$$L_X \gamma^{ab} = 0, \qquad L_X \theta_a = 0 \tag{1}$$

form an infinite-dimensional Lie algebra called the *Coriolis* algebra $\mathfrak{cor}(M, \gamma, \theta)$. Notably enough, these vector fields do not Lie-transport the Newtonian connection; nevertheless, a somewhat tedious calculation, using the above mentioned prescribed symmetry of the curvature, shows that

$$L_X \Gamma^{abc} = 0 \tag{1'}$$

where $\Gamma^{abc} \equiv \gamma^{ak}\gamma^{b\ell}\Gamma_{k\ell}{}^{c}$.

In the standard case one finds

$$X^A = \omega(t)^A_B\,x^B + \varrho(t)^A, \qquad\qquad X^0 = \tau \tag{2}$$

where ω (resp. ϱ) is some $\mathfrak{so}(n)$ (resp. $\mathbf{R}^n$) valued function of time $t = x^0$ and $\tau \in \mathbf{R}$ an infinitesimal time translation. Coriolis vector fields generate the so called 'accelerated frames' transformations.

ii) *Galilei algebra*

The affine Coriolis vector fields X, *viz*

$$L_X \gamma^{ab} = 0, \qquad L_X \theta_a = 0, \qquad L_X \Gamma_{ab}{}^{c} = 0 \tag{3}$$

form the *Galilei* Lie algebra $\mathfrak{gal}(M, \gamma, \theta, \Gamma)$ of our Newton-Cartan structure. This algebra has maximal dimension $(n+1)(n+2)/2$.

In the flat case we obtain

$$X^A = \omega^A_B \, x^B + \beta^A \, t + \sigma^A, \qquad X^0 = \tau \tag{4}$$

with $(\omega, \beta) \in \mathfrak{se}(n)$ and $(\sigma, \tau) \in \mathbf{R}^{n+1}$.

This general definition, originally due to Trautman [**Tra**], provides a clearcut geometrical status for the fundamental symmetries of Galilean classical mechanics and field theory.

iii) *Milne algebra*

Interestingly, there exists a less familiar intermediate algebra, namely the infinite-dimensional Lie algebra of those vector fields X such that

$$L_X \gamma^{ab} = 0, \qquad L_X \theta_a = 0, \qquad L_X \Gamma_a{}^{bc} = 0 \tag{5}$$

where $\Gamma_a{}^{bc} \equiv \gamma^{bk} \Gamma_{ak}{}^{c}$. We will call it the *Milne* algebra $\mathfrak{mil}(M, \gamma, \theta, \Gamma)$.

In the standard case (which encompasses Newton-Milne cosmology [**Mi**]) we get

$$X^A = \omega^A_B \, x^B + \varrho(t)^A, \qquad X^0 = \tau \tag{6}$$

with the same notation as before.

These vector fields constitute a Lie algebra corresponding to the infinitesimal Milne transformations introduced in [**CK**] which, indeed, admit the intrinsic definition given by Eqs (5) for a general Newton-Cartan structure $(M, \gamma, \theta, \Gamma)$.

2. Extending the Coriolis, Milne and Galilei algebras

It has been established [**Kü**] that any newtonian connection can be affinely decomposed according to $\Gamma_{ab}{}^{c} = {}^{U}\Gamma_{ab}{}^{c} + \theta_{(a}F_{b)k}\gamma^{kc}$ where [**Trü**]

$$ {}^{U}\Gamma_{ab}{}^{c} = \gamma^{ck}\Big(\partial_{(a}{}^{U}\gamma_{b)k} - \tfrac{1}{2}\partial_{k}{}^{U}\gamma_{ab}\Big) + \partial_{(a}\theta_{b)}\,U^{c} \tag{7} $$

is the unique NC connection for which the unit spacetime vector field U (i.e. $\theta_a U^a = 1$) is geodesic and curlfree, F being an otherwise arbitrary closed 2-form, locally

$$ F_{ab} = 2\,\partial_{[a}A_{b]}. \tag{8} $$

Here ${}^{U}\gamma$ is uniquely determined by ${}^{U}\gamma_{ak}\gamma^{kb} = \delta_a^b - U^b\theta_a$ and ${}^{U}\gamma_{ak}U^k = 0$.

As an illustration, standard NC structure corresponds to the gauge choice $U = \partial/\partial x^0$ and $A = -\phi\,\theta$.

So, NC strutures (M,γ,θ,Γ) are best represented by what we call *Newton-Cartan-Bargmann* (NCB) structures (M,γ,θ,U,A) via the previous formulæ. Now, experience suggests to think of a NCB structure as the sextuple $(M,\gamma,\theta,U,V,\phi)$ where

$$ V^a = U^a - \gamma^{ak}A_k \tag{9} $$

is an observer in rotation with respect to the 'ether' U [**CK**], while

$$ \phi = \tfrac{1}{2}\gamma^{k\ell}A_kA_\ell - A_kU^k \tag{10} $$

is the Newtonian potential relative to V. Conversely, $A_b = -{}^{U}\gamma_{bk}V^k + \Big(\phi - \tfrac{1}{2}{}^{U}\gamma_{k\ell}V^kV^\ell\Big)\theta_b$.

It must be emphasized that the 'principle of general covariance' has been be consistently adapted from general relativity to the NCB framework [**DK**], the specific Newtonian gauge group $G = \mathrm{Diff}(M) \circledS (\Omega^1(M)\times C^\infty(M))$ acting upon NCB structures according to

$$ \begin{pmatrix} \gamma \\ \theta \\ U \\ V \\ \phi \end{pmatrix} \longmapsto \mathcal{A}_* \begin{pmatrix} \gamma \\ \theta \\ U + \gamma(\Psi) \\ V + \gamma(d\mathcal{F}) \\ \phi + V(\mathcal{F}) + \tfrac{1}{2}\gamma(d\mathcal{F}, d\mathcal{F}) \end{pmatrix} \tag{11} $$

where $\mathcal{A}_*$ is the push-forward by $\mathcal{A} \in \mathrm{Diff}(M)$, also $\Psi \in \Omega^1(M)$ is a Galilei boost 1-form and $\mathcal{F} \in C^\infty(M)$ a special Newtonian gauge. As expected, G does act on NC structures via $\mathrm{Diff}(M)$ only

$$\begin{pmatrix} \gamma \\ \theta \\ \Gamma \end{pmatrix} \longmapsto \mathcal{A}_* \begin{pmatrix} \gamma \\ \theta \\ \Gamma \end{pmatrix}. \tag{12}$$

The infinitesimal action of the gauge group G on a NCB structure thus reads

$$\delta \begin{pmatrix} \gamma \\ \theta \\ U \\ V \\ \phi \end{pmatrix} = \begin{pmatrix} L_X\gamma \\ L_X\theta \\ L_X U + \gamma(\psi) \\ L_X V + \gamma(df) \\ X(\phi) + V(f) \end{pmatrix} \tag{13}$$

where $X \in \mathrm{Vect}(M)$, $\psi \in \Omega^1(M)$ and $f \in C^\infty(M)$. The associated Lie algebra structure is explicitly given by

$$\Big[(X,\psi,f),(X',\psi',f')\Big] = \Big([X,X'], L_X\psi' - L_{X'}\psi, X(f') - X'(f)\Big). \tag{14}$$

i) *The extended Coriolis algebra*

By looking at the infinitesimal NCB gauge transformations such that

$$\delta\gamma = 0, \qquad \delta\theta = 0, \qquad \delta U = 0 \tag{15}$$

we readily find a Lie algebra denoted by $\widetilde{\mathfrak{cor}}(M,\gamma,\theta,U)$ that clearly consists of couples $(X,f) \in \mathfrak{cor}(M,\gamma,\theta) \times C^\infty(M)$ —the boosts are already fixed (modulo θ): $\psi = {}^U\gamma([U,X])$. The Lie brackets, inherited from Eq. (14), reduce hence to

$$\Big[(X,f),(X',f')\Big] = \Big([X,X'], X(f') - X'(f)\Big) \tag{16}$$

and yield the semidirect product structure

$$\widetilde{\mathfrak{cor}}(M,\gamma,\theta,U) = \mathfrak{cor}(M,\gamma,\theta) \,ⓢ\, C^\infty(M). \tag{17}$$

ii) *The extended Milne algebra*

Likewise, the Lie algebra $\widetilde{\mathfrak{mil}}(M,\gamma,\theta,U,V)$ of all gauge transformations (13) such that

$$\delta\gamma = 0, \qquad \delta\theta = 0, \qquad \delta U = 0, \qquad \delta V = 0 \tag{18}$$

is formed of pairs (X,ξ) with $X \in \mathfrak{mil}(M,\gamma,\theta,\Gamma)$ and $\xi \in C^\infty(T)$ where $T \equiv M/\ker(\theta)$ is the canonical time axis —the general solution of $\gamma(df) = [V,X]$ being of the form $f = \xi + f_X$ with f_X uniquely determined by the condition $f_0 = 0$. The Lie brackets

$$\Big[(X,\xi),(X',\xi')\Big] = \Big([X,X'], X(\xi' + f_{X'}) - X'(\xi + f_X) - f_{[X,X']}\Big) \tag{19}$$

therefore lead to the following *non central extension*

$$0 \to C^\infty(T) \to \widetilde{\mathfrak{mil}}(M,\gamma,\theta,U,V) \to \mathfrak{mil}(M,\gamma,\theta,\Gamma) \to 0. \tag{20}$$

In the standard case and with the notation of section 1, the Lie brackets $(X'',\xi'') = \Big[(X,\xi),(X',\xi')\Big]$ read

$$\begin{cases} \omega'' = \omega'\omega - \omega\omega' \\ \varrho'' = \omega'\varrho - \omega\varrho' + \tau\dot{\varrho}' - \tau'\dot{\varrho} \\ \tau'' = 0 \\ \xi'' = \tau\dot{\xi}' - \tau'\dot{\xi} + \varrho\cdot\dot{\varrho}' - \varrho'\cdot\dot{\varrho} \end{cases} \tag{21}$$

with $\omega \in \mathfrak{so}(n)$, $\varrho \in C^\infty(T,\mathbf{R}^n)$, $\tau \in \mathbf{R}$ and $\xi \in C^\infty(T)$.

iii) *The extended Galilei algebra*

At last $\widetilde{\mathfrak{gal}}(M,\gamma,\theta,U,V,\phi)$ defined as the stabilizer of a NCB structure, *viz*

$$\delta\gamma = 0, \qquad \delta\theta = 0, \qquad \delta U = 0, \qquad \delta V = 0, \qquad \delta\phi = 0 \tag{22}$$

consists of pairs (X,ξ) with $X \in \mathfrak{gal}(M,\gamma,\theta,\Gamma)$ and $\xi \in \mathbf{R}$ (the overall additive constant in the solution f of $df = (-X(\phi) + {}^U\gamma(L_XV,V))\theta - {}^U\gamma(L_XV))$. The Lie brackets given by Eq. (19) still hold and, this time, we find a non trivial finite dimensional *central extension*

$$0 \to \mathbf{R} \to \widetilde{\mathfrak{gal}}(M,\gamma,\theta,U,V,\phi) \to \mathfrak{gal}(M,\gamma,\theta,\Gamma) \to 0. \tag{23}$$

In the case of flat spacetime we get, with the same notation as before, the centrally extended Galilei algebra

$$\tag{24} \begin{cases} \omega'' = \omega'\omega - \omega\omega' \\ \beta'' = \omega'\beta - \omega\beta' \\ \sigma'' = \omega'\sigma - \omega\sigma' + \beta'\tau - \beta\tau' \\ \tau'' = 0 \\ \xi'' = \sigma \cdot \beta' - \sigma' \cdot \beta \end{cases}$$

i.e. the Lie algebra of the *Bargmann group*.

Acknowledgements : Discussions with B. Carter and J.-M. Souriau are acknowledged. Thanks are due to C. Barrabès and P. Horváthy for valuable help.

References

[**CK**] B. Carter and I.M. Khalatnikov, *Canonically Covariant Formulation of Landau's Newtonian Superfluid Dynamics*, Preprint DARC, Meudon (1992).

[**Ca**] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisé*, Ann. Sci. École Norm. Sup. (4) **40** (1923), 325–412.

[**Kü**] H.P. Künzle, *Galilei and Lorentz structures on spacetime: Comparison of the corresponding geometry and physics*, Ann. Inst. H. Poincaré. Phys. Théor. **17** (1972), 337–362.

[**DK**] C. Duval and H.P. Künzle, *Sur les connexions newtoniennes et l'extension non triviale du groupe de Galilée*, C.R. Acad. Sci. Paris **285 A** (1977), 813–816; *Dynamics of continua and particles from general covariance of Newtonian gravitation theory*, Rep. Math. Phys. **13** (1978), 351–368; *Minimal Gravitational Coupling in the Newtonian Theory and the Covariant Schrödinger Equation*, Gen. Rel. Grav. **16** (1984), 333–347.

[**Mi**] E.A. Milne, *A Newtonian Expanding Universe*, Quat. J. Math. (Oxford Ser.) **5** (1934), 64–72.

[**Tra**] A. Trautman, *Sur la théorie newtonienne de la gravitation*, C.R. Acad. Sci. Paris **257** (1963), 617–620; *Comparison of Newtonian and relativistic theories of space time*, pp. 413–425 in Perspectives in Geometry and Relativity (B. Hoffmann, ed.), Indiana University Press, Bloomington, 1964.

[**Trü**] M. Trümper, *Lagrangian Mechanics and the Geometry of Configuration Spacetime*, Ann. Phys. (N.Y.) **149** (1983), 203–233.